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Two-matrix string model as constrained (2+1)-dimensional integrable system

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Abstract

We show that the 2-matrix string model corresponds to a coupled system of $2+1$-dimensional KP and modified KP ($(\mathrm{m})\mathrm{KP}_{2+1}$) integrable equations subject to a specific "symmetry" constraint. The latter together with the Miura-Konopelchenko map for $(\mathrm{m})\mathrm{KP}_{2+1}$ are the continuum incarnation of the matrix string equation. The $(\mathrm{m})\mathrm{KP}_{2+1}$ Miura and Bäcklund transformations are natural consequences of the underlying lattice structure. The constrained $(\mathrm{m})\mathrm{KP}_{2+1}$ system is equivalent to a $1+1$-dimensional generalized KP-KdV hierarchy related to graded SL(3,1). We provide an explicit representation of this hierarchy, including the associated W(2,1)-algebra of the second Hamiltonian structure, in terms of free currents.

1. Introduction

One of the remarkable features of integrable Toda lattice hierarchies, describing the multi-matrix string models [1], is the existence of a natural elegant way [2] to extract from them continuum differential integrable systems without taking any continuum (double-scaling) limit. These systems are of generalized KP-KdV type and are associated with one fixed lattice site. This proves to be an efficient approach for exact calculations of multi-matrix model correlation functions and exhibits deep relations with topological field theory. In particular, the two-matrix model in the formalism of [2] provides a systematic unified framework for exact treatment via integrable hierarchies of $c = 1$ string theory (for related approaches, cf. [3]). The principal instrument are

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the $W_{1+\infty}$ constraints on the two-matrix model partition function, which are the facets of the pertinent string equation.

The aim of the present note is to construct a differential integrable system out of the Toda lattice hierarchy, underlying the two-matrix model, which directly incorporates in itself the whole information of the string equation, rather than imposing it as an infinite number of constraints on the discrete model partition function. We achieve this by assuming that at least one of the matrix potentials in the two-matrix model is of *finite* order p. In the simplest nontrivial $p = 3$ case the continuum system, we are looking for, turns out to be a coupled system of $2 + 1$-dimensional KP and modified KP ($(\mathrm{m})\mathrm{KP}_{2+1}$) integrable equations subject to two additional constraints. The latter are identified as the known Miura-Konopelchenko map [4] and a specific KP "symmetry" constraint [5]. We demonstrate that these two differential constraints embody the matrix string equation. In particular, the two-matrix model "susceptibility" satisfies the above string-constrained KP_{2+1} equation.

Lattice translations between different sites in the context of Toda hierarchy are shown to generate the well-known $(\mathrm{m})\mathrm{KP}_{2+1}$ Bäcklund transformations [6,7] which are canonical symmetry of the $(\mathrm{m})\mathrm{KP}_{2+1}$ Hamiltonian system. Furthermore, from the constrained $2 + 1$-dimensional $(\mathrm{m})\mathrm{KP}$ system we derive an equivalent *unconstrained* $1 + 1$-dimensional integrable system of generalized KP-KdV type related to graded $\mathrm{SL}(3,1)$ algebra [8,9]. Systems of this type were previously obtained in [10,11] by imposing *ad hoc* second-class constraints on multi-boson KP hierarchies. The corresponding second Hamiltonian structure is the nonlinear $\mathrm{W}(2,1)$ algebra, generalizing the $\hat{\mathrm{W}}_\infty$ algebra [12]. We provide an explicit free-field representation of $\mathrm{W}(2,1)$ (cf. also [13,9]).

2. Discrete integrable systems from two-matrix string models

This section starts by collecting some basic results on the discrete integrable systems – generalized Toda-like lattice systems, which are associated with multi-matrix string models [2]. The associated Toda matrices contain in general *finite* number of non-zero diagonals unlike the case of the full generalized Toda lattice hierarchy [14]. We present the relevant solutions for the Toda matrices in a form appropriate for establishing connection with $2 + 1$-dimensional integrable models.

Specifically, we shall consider the two-matrix model with partition function:

$$Z_N[t,\tilde{t},g] = \int dM_1 dM_2 \exp - \left\{ \sum_{r=1}^{p_1} t_r \operatorname{Tr} M_1^r + \sum_{s=1}^{p_2} \tilde{t}_s \operatorname{Tr} M_2^s + g \operatorname{Tr} M_1 M_2 \right\} \tag{1}$$

where $M_{1,2}$ are Hermitian $N \times N$ matrices, and the orders of the matrix "potentials" $p_{1,2}$ may be finite or infinite. In Ref. [2] it was shown that, by using the method of generalized orthogonal polynomials [15], the partition function (1) and its derivatives w.r.t. the parameters $(t_r, \tilde{t}_s, g)$ can be explicitly expressed in terms of solutions to generalized Toda-like lattice systems associated with (1). The corresponding linear problem and Lax (or "zero-curvature") representation of the latter read [2]:

$$Q_{nm}\psi_m = \lambda\psi_n\,, \qquad - g\bar{Q}_{nm}\psi_m = \frac{\partial}{\partial\lambda}\psi_n \tag{2}$$

$$\frac{\partial}{\partial t_r}\psi_n = \left(Q^r_{(+)}\right)_{nm}\psi_m\,, \quad \frac{\partial}{\partial \tilde{t}_s}\psi_n = -\left(\bar{Q}^s_{-}\right)_{nm}\psi_m \tag{3}$$

$$\frac{\partial}{\partial t_r}Q = \left[Q^r_{(+)}, Q\right]\,, \quad \frac{\partial}{\partial \tilde{t}_s}Q = \left[Q, \bar{Q}^s_{-}\right] \tag{4}$$

$$g\left[Q, \bar{Q}\right] = \mathbb{1} \tag{5}$$

$$\frac{\partial}{\partial t_r}\bar{Q} = \left[Q^r_{(+)}, \bar{Q}\right]\,, \quad \frac{\partial}{\partial \tilde{t}_s}\bar{Q} = \left[\bar{Q}, \bar{Q}^s_{-}\right] \tag{6}$$

The subscripts $-/+$ denote lower/upper triangular parts, whereas $(+)/(-)$ denote upper/lower triangular plus diagonal parts. The parametrization of the matrices Q and $\bar{Q}$ is as follows:

$$Q_{nn} = a_0(n)\,, \quad Q_{n,n+1} = 1\,, \quad Q_{n,n-k} = a_k(n) \quad k = 1,\ldots,p_2 - 1$$
$$Q_{nm} = 0 \quad \text{for} \quad m - n \geq 2\ , \quad n - m \geq p_2 \tag{7}$$
$$\bar{Q}_{nn} = b_0(n)\,, \quad \bar{Q}_{n,n-1} = R_n\,, \quad \bar{Q}_{n,n+k} = b_k(n) R^{-1}_{n+1} \ldots R^{-1}_{n+k} \quad k = 1,\ldots,p_1 - 1$$
$$\bar{Q}_{nm} = 0 \quad \text{for} \quad n - m \geq 2\ , \quad m - n \geq p_1 \tag{8}$$

From now on we shall consider the first evolution parameters $t_1, \tilde{t}_1$ as coordinates of a two-dimensional space, i.e., $\tilde{t}_1 \equiv x$ and $t_1 \equiv y$, so all modes $a_k(n), b_k(n)$ and R_n depend on $(x, y; t_2, \ldots, t_{p_1}; \tilde{t}_2, \ldots, \tilde{t}_{p_2})$.

The second lattice equation of motion (6) for $s = 1$, using parametrization (8), gives:

$$\partial_x R_n = R_n \left(b_0(n) - b_0(n-1) \right)\,, \quad \partial_x b_0(n) = b_1(n) - b_1(n-1) \tag{9}$$
$$\partial_x \left(\frac{b_k(n)}{R_{n+1} \ldots R_{n+k}} \right) = \frac{b_{k+1}(n) - b_{k+1}(n-1)}{R_{n+1} \ldots R_{n+k}}\,, \quad k \geq 2 \tag{10}$$

Similarly, the first lattice equation of motion (6) for $r = 1$ gives:

$$\partial_y b_0(n) = R_{n+1} - R_n\,, \quad \partial_y b_k(n) = R_{n+1} b_{k-1}(n+1) - R_{n+k} b_{k-1}(n) \tag{11}$$

for $k \geq 1$. From the above equations one can express all $b_k(n \pm \ell)$, $k \geq 2$ and $R_{n\pm\ell}$ (ℓ – arbitrary integer) as functionals of $b_0(n), b_1(n)$ at a *fixed* lattice site n.

In complete analogy, the lattice equations of motion (4) for $r = 1, s = 1$ read explicitly:

$$\partial_x a_0(n) = R_{n+1} - R_n\,, \quad \partial_x a_k(n) = R_{n-k+1} a_{k-1}(n) - R_n a_{k-1}(n-1) \tag{12}$$

(with $k \geq 1$) and

$$\partial_y a_0(n) = a_1(n+1) - a_1(n)\,, \quad \partial_y \left(\frac{a_k(n)}{R_n \ldots R_{n-k+1}} \right) = \frac{a_{k+1}(n+1) - a_{k+1}(n)}{R_n \ldots R_{n-k+1}} \tag{13}$$

with $k \geq 1$. Following (9), (11), (12) we obtain the "duality" relations:

$$\partial_y b_1(n) = \partial_x R_{n+1}\,, \quad \partial_x a_0(n) = \partial_y b_0(n)\,, \quad \partial_x a_1(n) = \partial_y R_n \tag{14}$$

From the above one gets the two-dimensional Toda lattice equation:

$$\partial_y \ln R_n = a_0(n) - a_0(n-1) \quad \longrightarrow \quad \partial_x \partial_y \ln R_n = R_{n+1} - 2R_n + R_{n-1} \tag{15}$$

Eqs. (12)–(15) allow to express all $a_k(n \pm \ell)$, $k \geq 1$ and $R_{n\pm\ell}$ as functionals of $a_0(n)$ and R_n (or $a_1(n)$ instead) at a fixed lattice site n. Furthermore, due to Eqs. (12) and (11) for $k = 1$, all matrix elements of Q and $\bar{Q}$ are functionals of $b_0(n), b_1(n)$ at a fixed lattice site n. Alternatively, due to (14) we can consider $a_0(n)$ and R_{n+1} as independent functions instead of $b_0(n), b_1(n)$.

Let us also add the explicit expressions for the flow equations for $b_0(n), b_1(n), R_{n+1}$ resulting from (4) and (6):

$$\frac{\partial}{\partial \tilde{t}_s} b_0(n) = \partial_x \left(\bar{Q}^s \right)_{nn}\,, \quad \frac{\partial}{\partial \tilde{t}_s} b_1(n) = \partial_x \left[R_{n+1} \left(\bar{Q}^s \right)_{n,n+1} \right]\,, \quad \frac{\partial}{\partial \tilde{t}_s} R_{n+1} = \partial_x \left(\bar{Q}^s \right)_{n+1,n} \tag{16}$$
$$\frac{\partial}{\partial t_r} b_0(n) = \partial_x \left(Q^r \right)_{nn}\,, \quad \frac{\partial}{\partial t_r} b_1(n) = \partial_x \left[R_{n+1} \left(Q^r \right)_{n,n+1} \right]\,, \quad \frac{\partial}{\partial t_r} R_{n+1} = \partial_x \left(Q^r \right)_{n+1,n} \tag{17}$$

3. Solution of string equation for finite-order matrix potentials

From now on we assume that one of the matrix potentials in (1), e.g., the second one has a finite order p_2, which implies that the matrix Q has a finite number of diagonals below the main diagonal (cf. (7)). In this case the lattice equations of motion impose additional constraints relating the two independent functions $b_0(n)$ and $b_1(n)$ (or R_{n+1}).

More precisely, we find that for $p_2 < \infty$ the lower triangular plus diagonal part of Q is expressed through $\bar{Q}$ on the space of solutions to (12)–(15) in the following form (in what follows, we shall ignore arbitrary integration constants without loss of generality):

$$Q = \bar{Q}^{p_2-1}_{(-)} + \left(\frac{1}{g}x\right)\mathbb{1} + I_+ \tag{18}$$

where g is the two-matrix model coupling parameter appearing in the string equation (5), and $I_{+nm} = \delta_{n+1,m}$. In particular, for the lowest non-zero diagonal of Q one has:

$$a_{p_2-1}(n) = R_n \dots R_{n-p_2+2} = \bar{Q}^{p_2-1}_{n,n-(p_2-1)} \tag{19}$$

which follows from the lattice equation (13) for $k = p_2 - 1$ and from the explicit parametrization of $\bar{Q}$ (8).

Eq. (18) can be proved by induction w.r.t. $k = p_2-1, \dots, 1, 0$ starting from (19), upon comparing Eqs. (12)–(13) with the component form of the matrix equations of motion $\partial_y \bar{Q}^s = \left[Q_{(+)}, \bar{Q}^s\right]$ and $\partial_x \bar{Q}^s = \left[\bar{Q}^s, \bar{Q}_-\right]$ for arbitrary integer power s.

The remaining lattice equations of motion – the first equations of (12) and (15), imply the following two additional constraints on the independent functions $b_0(n)$ and R_{n+1} (or $b_1(n)$)

$$\partial_y b_0(n) = \frac{1}{g} + \partial_x\left(\bar{Q}^{p_2-1}_{nn}\right), \quad \partial_y R_{n+1} = \partial_x\left(\bar{Q}^{p_2-1}_{n+1,n}\right) \tag{20}$$

In fact, Eqs. (20) are nothing but the component form of the string equation (5). Indeed, it can be rewritten in the form (upon using (18) and (4), (6)): $(\partial_y - \partial/\partial\tilde{t}_{p_2-1})\,\bar{Q} = \frac{1}{g}\mathbb{1}$, i.e.

$$\frac{\partial}{\partial\tilde{t}_{p_2-1}} b_0(n) = \partial_y b_0(n) - \frac{1}{g}, \quad \frac{\partial}{\partial\tilde{t}_{p_2-1}} R_{n+1} = \partial_y R_{n+1}, \quad \frac{\partial}{\partial\tilde{t}_{p_2-1}} b_k(n) = \partial_y b_k(n) \tag{21}$$

for $k \geq 1$. Now, inserting (16) into (20) we find that the latter two equations precisely coincide with the nn and $n+1, n$ component of the string equation (21). In particular, the evolution parameter $\tilde{t}_{p_2-1}$ is not independent but rather essentially coincides with $y \equiv t_1$. It turns out in what follows that it is more convenient to use $\bar{y} \equiv \tilde{t}_{p_2-1}$ as a second space coordinate instead of y.

To conclude this section, let us note that there is a complete duality under $p_1 \longleftrightarrow p_2$ when the order p_1 of the first matrix potential in (1) is also finite. We obtain the exact analogs of Eqs. (18), (20) and (21) by interchanging $p_1 \longleftrightarrow p_2$, $x \equiv \tilde{t}_1 \longleftrightarrow t_1 \equiv y$, $\tilde{t}_{p_2-1} \longleftrightarrow t_{p_1-1}$, $Q_{(-)} \longleftrightarrow \bar{Q}_{(+)}$.

4. Derivation of $\mathbf{mKP_{2+1}}$ and $\mathbf{KP_{2+1}}$ from lattice systems

Using the parametrization (7)–(8), the equations of the auxiliary linear Lax problem (2), (3) acquire the form:

$$\lambda\psi_n = \psi_{n+1} + a_0(n)\psi_n + \sum_{k=1}^{p_2-1} a_k(n)\psi_{n-k} \tag{22}$$

$$-\frac{1}{g}\frac{\partial}{\partial\lambda}\psi_n = R_n\psi_{n-1} + b_0(n)\psi_n + \sum_{k=1}^{p_1-1}\frac{b_k(n)}{R_{n+1}\ldots R_{n+k}}\psi_{n+k} \tag{23}$$

$$\partial_x\psi_n = -R_n\psi_{n-1}\,, \quad \partial_y\psi_n = \psi_{n+1} + a_0(n)\psi_n \tag{24}$$

In particular, one has from the first equation (24):

$$\psi_{n-k} = \frac{(-1)^k}{R_{n-k+1}}\partial_x\frac{1}{R_{n-k+2}}\partial_x\ldots\frac{1}{R_n}\partial_x\psi_n \tag{25}$$

Inserting (24)-(25) into (22) one obtains the following *two-dimensional* differential Lax spectral equation for ψ_n at a *fixed* lattice site n[7]:

$$L_{p_2-1}(n)\psi_n = 0 \tag{26}$$

$$\begin{aligned} L_{p_2-1}(n) &= \partial_{\bar{y}} + \sum_{k=1}^{p_2-1}\frac{(-1)^k a_k(n)}{R_n\ldots R_{n-k+1}}\left(\partial_x - \partial_x\ln\left(R_n\ldots R_{n-k+2}\right)\right)\cdots\left(\partial_x - \partial_x\ln R_n\right)\partial_x \\ &= \partial_{\bar{y}} + (-1)^{p_2-1}\partial_x^{p_2-1} + \sum_{k=1}^{p_2-2}(-1)^k f_k^{(p_2-1)}\big(b_0(n), b_1(n)\big)\partial_x^k \equiv \partial_y + M_{p_2-1}(n) \end{aligned} \tag{27}$$

where in the second line of Eq. (27) we have used the lattice equations of motion (18), i.e., $Q_- = \left(\bar{Q}^{p_2-1}\right)_-$, to express the coefficient functions of the 2-dimensional Lax differential operator in terms of the independent $b_{0,1}(n)$.

Similarly, the lattice flow equations w.r.t. $\tilde{t}_s$ (3) can be equivalently written as differential flow equations:

$$\frac{\partial}{\partial\tilde{t}_s}\psi_n = -M_s(n)\psi_n\,, \quad M_s(n) \equiv (-1)^s\partial_x^s + \sum_{k=1}^{s-1}(-1)^k f_k^{(s)}\big(b_0(n), b_1(n)\big)\partial_x^k \tag{28}$$

with $s = 2,\ldots,p_2$. Explicitly we have for the coefficient functions:

$$f_{s-1}^{(s)} = sb_0(n)\,, \quad f_{s-2}^{(s)} = sb_1(n) + \binom{s}{2}\left(b_0^2(n) - \partial_x b_0(n)\right)\,, \quad \text{etc.} \tag{29}$$

In particular, comparing (28)-(29) and (26)-(27), we note that the Lax spectral equation (26) is in fact the (p_2-1)-th flow equation (28).

The compatibility conditions for (28):

$$\frac{\partial}{\partial\tilde{t}_r}M_s(n) - \frac{\partial}{\partial\tilde{t}_s}M_r(n) + \big[M_r(n), M_s(n)\big] = 0\,, \quad r,s = 2,\ldots p_2 \tag{30}$$

yield for $r = p_2 - 1$ (i.e., $\tilde{t}_{p_2-1} = \bar{y}$) a system of $2+1$-dimensional integrable nonlinear evolution equations for $b_0(n), b_1(n)$ subject to the additional string-equation constraints (20). Let us consider in more detail the simplest nontrivial case $p_2 = 3$ where:

$$M_2(n) = \partial_x^2 - 2b_0(n)\partial_x\,, \quad M_3(n) = -\partial_x^3 + 3b_0(n)\partial_x^2 - 3\left(b_1(n) + b_0^2(n) - \partial_x b_0(n)\right)\partial_x \tag{31}$$

[7] In Eq. (26) we have hidden the spectral parameter λ by using the identity $\frac{\partial}{\partial\tilde{t}_{p_2-1}}\psi_n \equiv \partial_{\bar{y}}\psi_n = (\partial_y - \lambda)\psi_n$, which follows from (2), (3) and (18).

Eq. (30) with $r = 2, s = 3$ yields firstly the non-dynamical equation:

$$\partial_{\bar{y}} b_0(n) = \partial_x \left(2b_1(n) - \partial_x b_0(n) + b_0^2(n)\right) \tag{32}$$

which coincides with the first string equation (20) (for $p_2 = 3$). Secondly, it yields an evolution equation for $b_0(n)$:

$$4\frac{\partial}{\partial \tilde{t}_3} b_0(n) = \partial_x^3 b_0(n) - 2\partial_x b_0^3(n) + 3\partial_{\bar{y}} \bar{a}_0(n) + 6\bar{a}_0(n)\partial_x b_0(n) \tag{33}$$

$$\partial_x \bar{a}_0(n) = \partial_{\bar{y}} b_0(n), \quad \bar{a}_0(n) \equiv a_0(n) - \frac{1}{g}x = 2b_1(n) - \partial_x b_0(n) + b_0^2(n) \tag{34}$$

where we have used (14) and (32). Combining (32) with (33) yields an evolution equation for $b_1(n)$:

$$4\frac{\partial}{\partial \tilde{t}_3} b_1(n) = \partial_x^3 b_1(n) + 6\partial_x b_1^2(n) + 3\partial_{\bar{y}} R_{n+1}, \quad \partial_x R_{n+1} = \partial_{\bar{y}} b_1(n) \tag{35}$$

Finally, the second string equation (20) (for $p_2 = 3$) yields:

$$\partial_{\bar{y}} R_{n+1} = \partial_x \left[\partial_x R_{n+1} + 2b_0(n) R_{n+1}\right] \equiv M_2^*(n) R_{n+1} \tag{36}$$

where $M_2^*(n)$ denotes the operator adjoint to $M_2(n)$ in (31).

Now, it is straightforward to recognize Eq. (35) as the 2+1-dimensional KP equation (KP_{2+1}), Eq. (33) – as the modified KP equation (mKP_{2+1}), and Eq. (32) – as the associated 2+1-dimensional Miura-Konopelchenko map [4] relating them. However, the present two-matrix model derivation reveals an *additional* constraint (36) relating KP_{2+1} and mKP_{2+1} equations. This constraint can be identified as a particular example of a "symmetry constraint" [5] within the framework of the standard 1 + 1-dimensional KP hierarchy (KP_{1+1}) formalism. The KP reduction, triggered by (36), is studied in Section 6.

As demonstrated above, both the (m)KP_{2+1} equations (33) and (35), as well as the Miura-Konopelchenko map (32) and the "symmetry constraint" (36) KP reduction naturally arise from the underlying Toda-like lattice integrable structure associated with the two-matrix string model. Let us emphasize that the Miura-Konopelchenko map (32) and the "symmetry constraint" (36) explicitly embody the "string equation" (5) of the two-matrix model in the case $p_2 = 3$. Thus, they are the 2 + 1-dimensional analog of the $W_{1+\infty}$-constraints on the partition function (1) in the formalism of Refs. [2].

Through the method of orthogonal polynomials the two-matrix model partition function (1) is given by:

$$Z_N[t, \tilde{t}, g] = \text{const}\, N! \prod_{n=0}^{N-1} h_n, \quad h_n = \exp \int^x b_0(n) \tag{37}$$

Thus, its calculation reduces to finding the appropriate solutions of (m)KP_{2+1} equations (33), (35) subject to the "string equation" constraint (36). In particular, as follows from (37) and (9), the "susceptibility" $\partial_x^2 \ln Z_N = b_1(N-1)$ satisfies the string-constrained KP_{2+1} equation (35), (36).

5. Symmetries of constrained KP_{2+1} and mKP_{2+1}

In the previous section we represented the differential KP_{2+1} and mKP_{2+1} integrable systems explicitly in terms of objects from the underlying generalized Toda-like lattice structure. Now we are going to show that this underlying lattice structure naturally exhibits the discrete Bäcklund transformations for KP_{2+1} and mKP_{2+1}.

On the lattice itself we have two obvious discrete symmetry operations:

(a) lattice site translation $n \to n-1$, in particular, for the discrete Lax "wave" function

$$\psi_{n+1} \to \psi_n = -R_{n+1}^{-1}\partial_x\psi_{n+1} \tag{38}$$

(b) similarity (phase) transformations on the same lattice site, e.g., gauge transformations

$$\psi_n \to \widetilde{\psi}_n = e^{\phi_n}\psi_n \equiv \Phi_n\psi_n \tag{39}$$

First, we consider type (a) lattice symmetry (38). Going back to mKP_{2+1} (33) and KP_{2+1} (35), we observe that $b_0(n)$ and $b_1(n)$ do satisfy them *for any* fixed site n on the underlying lattice, i.e., if $b_0(n), b_1(n)$ satisfy (33), (35), so do $b_0(n-1), b_1(n-1)$. On the other hand, from the lattice equations of motion (9), (11) we obtain the following simple relations for the lattice shifts of the above functions:

$$b_0^{(-)}(n) \equiv b_0(n-1) = b_0(n) - \partial_x \ln(\partial_x\Phi_n) \quad , \quad b_1^{(-)}(n) \equiv b_1(n-1) = b_1(n) - \partial_x b_0(n) \tag{40}$$

$$\partial_x\Phi_n = R_n = R_{n+1} - \partial_{\bar{y}} b_0(n) - \frac{1}{g} \tag{41}$$

which can be viewed as discrete symmetry (Bäcklund) transformations for $(\mathrm{m})\mathrm{KP}_{2+1}$ (33) and (35). Obviously, type (a) Bäcklund transformation (40) preserves the $(\mathrm{m})\mathrm{KP}_{2+1}$ "symmetry" constraint (36). Note that the two-matrix model coupling constant g emerges here as a free Bäcklund parameter.

Let us also note, that the Bäcklund transformation for $b_1(n)$ (40) (taking into account (32)) is a canonical transformation for the 2+1-dimensional KP Hamiltonian system (cf. [16]), i.e., it leaves invariant the canonical Poisson brackets and the Hamiltonian (here there is no difference between y and $\bar{y}$):

$$\{\, b_1(n)(x,y) \,,\, b_1(n)(x',y') \,\} = \frac{1}{16}\partial_x\delta(x-x')\delta(y-y') = \{\, b_1(n+1)(x,y) \,,\, b_1(n+1)(x',y') \,\} \tag{42}$$

$$H(n) = 2\int dx\,dy\,\left[(\partial_x b_1(n))^2 - 4b_1^3(n) - 3R_{n+1}^2\right] = H(n+1)\,, \quad \partial_x R_{n+1} = \partial_y b_1(n) \tag{43}$$

In the framework of the standard KP_{1+1} hierarchy formalism with KP_{1+1} Lax operators:

$$L_{\mathrm{mKP}} = \partial_x + v + \sum_{k\geq 1} v_k\partial_x^{-k}, \quad L_{\mathrm{KP}} = \partial_x + \sum_{k\geq 1} u_k\partial_x^{-k} \tag{44}$$

Eqs. (40) have already been obtained [6,7] (identifying $v = b_0(n)$, $u_1 \equiv 2u = 2b_1(n)$) via the method of pseudo-differential operator gauge transformations and are known as the first type of auto-Bäcklund transformations for conventional $(\mathrm{m})\mathrm{KP}_{2+1}$. However, in the present two-matrix model realization of mKP_{2+1} and KP_{2+1} there is an additional important property of this auto-Bäcklund transformation – the additional constraint (41) expressing $\Phi \equiv \Phi_n$ as a specific explicit functional of the $(\mathrm{m})\mathrm{KP}_{2+1}$ functions u, v. This additional constraint on the auto-Bäcklund transformation results from the $(\mathrm{m})\mathrm{KP}_{2+1}$ "string-equation" constraint (36). Both constraints (36) and (41) are absent in the general $(\mathrm{m})\mathrm{KP}_{2+1}$ setting.

Next, we consider type (b) lattice symmetry (39). It must preserve the form of the 2-dimensional Lax and the corresponding $\tilde{t}_s$-flow generating operators:

$$\Phi_n^{-1}\left(\frac{\partial}{\partial\tilde{t}_s} + M_s(n)\right)(\Phi_n\cdot) = \frac{\partial}{\partial\tilde{t}_s} + \widetilde{M}_s(n)\,, \quad s = 2,\ldots p_2 \tag{45}$$

$$\widetilde{M}_s(n) \equiv (-1)^s\partial_x^s + \sum_{k=1}^{s-1}(-1)^k f_k^{(s)}\left(\widetilde{b}_0, \widetilde{b}_1\right)\partial_x^k \quad , \quad \widetilde{b}_0(n) = b_0(n) - \partial_x\ln\Phi_n \quad , \quad \widetilde{b}_1(n) = b_1(n) \tag{46}$$

where the last two equations represent type (b) Bäcklund transformation for unconstrained $(\mathrm{m})\mathrm{KP}_{2+1}$ [6,7]. Comparing (45) with the similarity relation between the operators $M_s(n)$ and $M_s(n+1)$ implied by type (a) lattice symmetry (38):

$$\frac{1}{R_{n+1}}\partial_x\left[\frac{\partial}{\partial\tilde{t}_s}+M_s(n+1)\right]\partial_x^{-1}\left(R_{n+1}\cdot\right)=\frac{\partial}{\partial\tilde{t}_s}+M_s(n)\,,\quad s=2,\ldots p_2 \tag{47}$$

we find that Φ_n in (45)-(46) is again given by Eq. (41). However, with such Φ_n type (b) Bäcklund transformation (46) *does not* preserve the string-equation constraint (36). Therefore, only type (a) (lattice translation) Bäcklund transformation (40) survives in the two-matrix-model realization of $(\mathrm{m})\mathrm{KP}_{2+1}$.

Concluding this section, let us consider also the continuum symmetries of the constrained $(\mathrm{m})\mathrm{KP}_{2+1}$. In this respect we note further important differences between the string matrix-model realization of mKP_{2+1} and KP_{2+1} integrable systems, on one hand, and the standard KP_{1+1} integrable hierarchy, on the other hand. In the present $2+1$-dimensional matrix-model formalism the number p_2 of the usual KP time-evolution parameters $(\tilde{t}_1=x,\tilde{t}_2,\ldots,\tilde{t}_{p_2-1}=\tilde{y},\tilde{t}_{p_2})$ is *finite*, e.g., $p_2=3$ in the above analysis, unlike the KP_{1+1} case. Furthermore, besides them there is also an additional set of time-evolution parameters $t_1\equiv y,t_2,\ldots,t_{p_1}$, which might be *infinite* in number, i.e., $p_1\to\infty$. Their corresponding flows, given by (17), are both commuting with the $\tilde{t}_s$-flows (given by (16)), as well as commuting among themselves. So the latter correspond to an (infinite-dimensional) Abelian symmetry algebra for the constrained string-matrix-model's $(\mathrm{m})\mathrm{KP}_{2+1}$ system.

The nature of the continuum symmetries corresponding to the t_r-flows could be better understood if one considers the constrained $(\mathrm{m})\mathrm{KP}_{2+1}$ in terms of the equivalent $1+1$ dimensional integrable system (Eqs. (48)-(52) below).

6. Equivalent 1 + 1-dimensional formulation of constrained $(\mathrm{m})\mathrm{KP}_{2+1}$

The string-constrained $(\mathrm{m})\mathrm{KP}_{2+1}$ system (33)-(36), which describes the two-matrix model (for $p_2=3$), can be explicitly reduced to an equivalent $1+1$-dimensional generalized KP-KdV integrable system. Upon excluding in (33) $\bar{a}_0(n)$ via (34) and (32), and substituting (36) into (35), the corresponding reduced $1+1$-dimensional integrable system can be written in the following form:

$$\frac{\partial}{\partial\tilde{t}_3}R_{n+1}=\partial_x\left[\partial_x^2R_{n+1}+3\left(b_1(n)+b_0^2(n)\right)R_{n+1}+3b_0(n)\partial_xR_{n+1}\right] \tag{48}$$

$$\frac{\partial}{\partial\tilde{t}_3}b_0(n)=\partial_x\left[\partial_x^2b_0(n)+b_0^3(n)+3b_1(n)b_0(n)+\frac{3}{2}R_{n+1}-\frac{3}{2}\partial_x\left(b_1(n)+b_0^2(n)\right)\right] \tag{49}$$

$$4\frac{\partial}{\partial\tilde{t}_3}b_1(n)=\partial_x\left[\partial_x^2b_1(n)+6b_1^2(n)+3\partial_xR_{n+1}+6b_0(n)R_{n+1}\right] \tag{50}$$

Similarly, the $\tilde{t}_2$ flow equations acquire the form:

$$\frac{\partial}{\partial\tilde{t}_2}R_{n+1}=\partial_x\left[\partial_xR_{n+1}+2b_0(n)R_{n+1}\right] \tag{51}$$

$$\frac{\partial}{\partial\tilde{t}_2}b_0(n)=\partial_x\left[2b_1(n)+b_0^2(n)-\partial_xb_0(n)\right]\,,\quad\frac{\partial}{\partial\tilde{t}_2}b_1(n)=\partial_xR_{n+1} \tag{52}$$

In complete analogy with the $2+1$-dimensional type (a) Bäcklund transformation (40), from the lattice equations of motion one immediately obtains the Bäcklund transformation for the $1+1$-dimensional system (48)-(52) resulting from negative lattice site translation:

$$b_0^{(-)}(n)\equiv b_0(n-1)=b_0(n)-\partial_x\ln\Big(R_{n+1}-\frac{1}{g}-\partial_x\left(2b_1(n)+b_0^2(n)-\partial_xb_0(n)\right)\Big) \tag{53}$$

$$b_1^{(-)}(n) \equiv b_1(n-1) = b_1(n) - \partial_x b_0(n) \tag{54}$$

$$R_{n+1}^{(-)} \equiv R_n = R_{n+1} - \frac{1}{g} - \partial_x \left(2b_1(n) + b_0^2(n) - \partial_x b_0(n)\right) \tag{55}$$

Again, as in (40), the two-matrix model coupling constant g appears as a free Bäcklund parameter.

The system (48)–(50) (or (51)–(52)) possesses Lax representation which is 1 + 1-dimensional analog of (26)–(27) (with $p_2 = 3$):

$$\lambda\psi_n = \widetilde{L}_{p_2-1}(n)\psi_n\,, \quad \frac{\partial}{\partial \widetilde{t}_s}\psi_n = -M_s(n)\psi_n\,, \quad s = 2,\ldots,p_2 \tag{56}$$

$$\widetilde{L}_{p_2-1}(n) = -\partial_x^{-1}\left(R_{n+1}\cdot\right) + \bar{Q}_{nn}^{p_2-1} + M_{p_2-1}(n) \tag{57}$$

$$\widetilde{L}_2(n) = -\partial_x^{-1}\left(R_{n+1}\cdot\right) + 2b_1(n) + \left(\partial_x - b_0(n)\right)^2\,, \quad \text{for } p_2 = 3 \tag{58}$$

In particular, one reproduces the lattice flow equations (16) as compatibility conditions for the linear auxiliary Lax problem (56). The systems (48)–(50) and (51)–(52) themselves arise as compatibility conditions:

$$\frac{\partial}{\partial \widetilde{t}_3}\widetilde{L}_2(n) + \left[M_3(n)\,,\,\widetilde{L}_2(n)\right] = 0\,, \quad \frac{\partial}{\partial \widetilde{t}_2}\widetilde{L}_2(n) + \left[M_2(n)\,,\,\widetilde{L}_2(n)\right] = 0 \tag{59}$$

with $\widetilde{L}_2(n)$ as in (58), and $M_{2,3}(n)$ as in (31).

Let us concentrate on the Lax operator (58) [8] (indication of the associated lattice site n is suppressed for brevity): $\widetilde{L}_2 = -D^{-1}\,R + 2b_1 + b_0^2 + \partial_x b_0 - D\ (2b_0) + D^2$. The Hamiltonian structures inherent in the integrable system (48)–(50) (or (51)–(52)) can naturally be identified using the general R-matrix scheme of Adler-Kostant-Symes/Reyman-Semenov-Tian-Shansky [17,18] in the context of the algebra of all pseudo-differential operators $\Psi\mathcal{DO}$. Namely, the first and the second Hamiltonian structures are given in terms of the following R-matrix Poisson brackets:

$$\{\,\langle L|X\rangle\,,\,\langle L|Y\rangle\,\}_R^{(1)} = -\langle L|\frac{1}{2}\,[\,RX\,,\,Y\,] + \frac{1}{2}\,[\,X\,,\,RY\,]\rangle \tag{60}$$

$$\{\,\langle L|X\rangle\,,\,\langle L|Y\rangle\,\}_R^{(2)} = -\langle L|\frac{1}{2}\,[\,R(XL)\,,\,Y\,] + \frac{1}{2}\,[\,X\,,\,R(LY)\,]\rangle \tag{61}$$

where L, X, Y are arbitrary elements of $\Psi\mathcal{DO}$: $L = \sum_{k\geq-\infty} D^k u_k$ and $X = \sum_{k\geq-\infty} X_k D^k$ and $<\cdot|\cdot>$ denotes the standard bilinear pairing via the Adler trace. $\langle L|X\rangle = \mathrm{Tr}_A(LX) = \int dx\,\mathrm{Res}\ (LX)$. Both Hamiltonian structures are compatible provided R is antisymmetric operator on $\Psi\mathcal{DO}$ w.r.t. the Adler trace. There are three different natural R-matrix operators on $\Psi\mathcal{DO}$ labelled by $\ell = 0, 1, 2$ [19]:

$$R_\ell = P_{\geq\ell} - P_{\leq\ell-1}\,, \quad P_{\geq\ell}X = \sum_{k\geq\ell} X_k D^k\,, \quad P_{\leq\ell-1}X = \sum_{k=\ell-1}^{\infty} X_{-k}D^{-k} \tag{62}$$

Now, one observes (cf. [20,21]) that the space of Lax operators $\widetilde{L}_2$ ($p_2 = 3$), and more generally – the Lax operators for arbitrary (finite) p_2 (57):

$$\widetilde{L}_{p_2-1} = -D^{-1}\,R + \sum_{k=0}^{p_2-2}(-1)^k D^k\,\widetilde{f}_k^{(p_2-1)} + (-1)^{p_2-1}D^{p_2-1} \tag{63}$$

[8] To avoid confusion, from here on D will denote the differential operator w.r.t. x, whereas $\partial_x f$ will denote derivative acting on a function. Note also the inverse ordering of differential operators and coefficient functions in the definition of the Lax operators.

with $\widetilde{f}_k^{(p_2-1)}$ simply expressed in terms of $f_k^{(p_2-1)}$ from (29) for $s = p_2 - 1$, span a *closed* R_1-coadjoint orbit on $\Psi\mathcal{D}\mathcal{O}$ with R_1 as in (62). The corresponding first Hamiltonian structure is given by the R-Kirillov-Kostant bracket (60) with $L = \widetilde{L}_2$ and $R = R_1 = P_{\geq 1} - P_{\leq 0}$. The $\tilde{t}_3$- and $\tilde{t}_2$-flow Eqs. (48)–(50) and (51)–(52) are Hamiltonian equations of motion w.r.t. the first bracket (60), which reads explicitly $\{ R(x) , b_0(x') \} = -\partial_x \delta(x-x')$ and $\{ b_1(x) , b_1(x') \} = -\frac{1}{2}\partial_x\delta(x-x')$, and with Hamiltonians[9] $H_N = -\frac{1}{N}\mathrm{Tr}_A \widetilde{L}_2^N$ for $N = \frac{5}{2}, 2$.

Concerning the construction of the second Hamiltonian structure, we note that R_1 from (62) is *not* antisymmetric. However, there exists a *symplectic* mapping [20,21,7] from the R_1-brackets to the standard R_0-brackets of the first KP Hamiltonian structure where now R_0 (62) is already antisymmetric:

$$\{ \langle \widetilde{L}_2 | X \rangle , \langle \widetilde{L}_2 | Y \rangle \}^{(1)}_{R_1} = \{ \langle \widetilde{L}_2^{(0)} | X \rangle , \langle \widetilde{L}_2^{(0)} | Y \rangle \}^{(1)}_{R_0} , \quad \widetilde{L}_2^{(0)} = e^{-\int_x b_0} \widetilde{L}_2 e^{\int_x b_0} \tag{64}$$

with $\widetilde{L}_2^{(0)} = -(D+b_0)^{-1} R + 2b_1 + D^2$. Therefore, we can now use the general formula (61) to construct the second Poisson brackets compatible with the first bracket, however, with a caution. Namely, let us emphasize the following important point. Eq. (61) is given for *generic* pseudo-differential Lax operators L. Restriction of L to arbitrary submanifold *does not* necessarily lead to a consistent restriction of the corresponding R-matrix Poisson brackets (61) (see, e.g. [22]). Thus, we have to prove that the restriction of generic L to $\widetilde{L}_2^{(0)}$ (64) is in fact a consistent Poisson reduction. This can be done by adapting the proof in [13] for the multi-boson reductions of the ordinary KP hierarchy. The final result for the second Hamiltonian structure for $\widetilde{L}_2^{(0)}$ reads[10]:

$$\begin{aligned} \{ \langle \widetilde{L}_2^{(0)} | X \rangle , \langle \widetilde{L}_2^{(0)} | Y \rangle \}^{(2)} &= \mathrm{Tr}_A \left((\widetilde{L}_2^{(0)} X)_{\geq 0} \widetilde{L}_2^{(0)} Y - (X \widetilde{L}_2^{(0)})_{\geq 0} Y \widetilde{L}_2^{(0)} \right) \\ &+ \frac{1}{2} \int dx \, \mathrm{Res}([\widetilde{L}_2^{(0)} , X]) \partial_x^{-1} \mathrm{Res}([\widetilde{L}_2^{(0)} , Y]) \end{aligned} \tag{65}$$

and the specific expressions for the nonlinear Poisson brackets among the coefficient fields can be found in Refs. [10,11,9].

It follows from the general considerations in [11], that the coefficient fields R, b_0, b_1 generate the W(2,1)-algebra, which is a generalization of the nonlinear $\hat{\mathrm{W}}_\infty$-algebra [12]. On the other hand, in [13] we succeeded to represent the usual multi-boson KP hierarchies in terms of canonical pairs of free fields abelianizing the second Hamiltonian structure of ordinary KP hierarchy. Using the latter result, we find a similar result for $\widetilde{L}_2^{(0)}$ – explicit expressions of the coefficient fields $(R_{n+1}, b_0(n), b_1(n))$ in terms of free fields (currents) (c_1, e_1, c_2):

$$\begin{aligned} R &= (\partial_x + e_1 + c_1 + 2c_2)(\partial_x + c_1) e_1 \\ b_1 &= \frac{1}{2}(\partial_x + c_1) e_1 - \frac{1}{2}(\partial_x + c_2) c_2 , \quad b_0 = e_1 + c_1 + c_2 \end{aligned} \tag{66}$$

$$\{ c_1(x) , e_1(x') \} = -\partial_x \delta(x-x') , \quad \{ c_2(x) , c_2(x') \} = \frac{1}{2}\partial_x \delta(x-x') , \quad \text{rest} = 0 \tag{67}$$

In turn, formulas (66), (67) provide an explicit free-field representation of W(2,1)-algebra embodied in the second bracket structure (65). Let us note, that (66)–(67) can also be obtained from the free current's realizations of the modified full KP hierarchy [8] by a specific Dirac constraint reduction. Similar free-field representations exist for $\mathrm{W}(p_2-1, 1)$ Poisson-bracket algebras for the higher Lax operators (63) which describe generalized graded $\mathrm{SL}(p_2, 1)$ KP-KdV integrable systems [8,9].

We conclude with a remark about the infinite-dimensional Abelian symmetry algebra of the $1+1$-dimensional integrable system (48)–(52), (58)–(59), generated by the t_r-flows (17). Inserting the solution for Q (18)

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[9] The overall minus sign in the definition of H_N is due to the inverse ordering of D's and coefficient functions in $\widetilde{L}_2$.

[10] The last term on the r.h.s. of (65) is a standard Dirac-bracket term due to the absence of next-to-leading differential order term in $\widetilde{L}_2^{(0)}$ (64).

into (17), we find that the corresponding vector fields $\frac{\partial}{\partial t_r}$ are Hamiltonian ones, with Hamiltonian functions $\mathcal{H}_r$ w.r.t. the first Poisson bracket (60) of the following form: $\mathcal{H}_r = \bar{H}_{r+1} = -\frac{1}{r+1}\mathrm{Tr}_A \bar{L}_2^{r+1}$ for integer $r \geq 1$. The Lax operator $\bar{L}_2$ is of the same form as (57)–(58), but with $a_0(n) = \bar{Q}_{nn}^2 + \frac{1}{g}x$ as a zero-order term instead of $\bar{Q}_{nn}^2 \equiv 2b_1(n) + b_0^2(n) - \partial_x b_0(n)$.

Outlook. It would be interesting to extend the analysis of Sections 4–6 beyond the simplest case of $p_2 = 3$ and for higher multi-matrix models. The general case of Lax operators $\widetilde{L}_{p_2-1}$ (63) with arbitrary finite differential part of order $p_2 - 1$ is studied in Ref. [9]. From Eq. (63) we expect then to find explicitly the generalized $\mathrm{SL}(p_2, q)$-KdV hierarchies with q restricted to $q = 1$. One recalls at this point that the generalized Kontsevich models [23] provide description of the continuous models (based on $c < 1$ minimal conformal models) only for $q = 1$ in the (p, q)-series and it would therefore be natural to study a possible relation.

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